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Factorization of the q-difference equation for continuous q-Jacobi polynomials

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Abstract
The formalism of factorization recently used by Atakishiyev *et al* (2007 *J. Phys. A: Math. Theor.* **40** 9311–7) to study the q-difference equation for continuous q-Hermite polynomials is extended to the continuous q-Jacobi polynomials. Particular cases of continuous q-Laguerre, q-Hermite, q-Legendre and q-ultraspherical polynomials are easily recovered.

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1. Introduction

An important and necessary step toward finding numerical solutions of the ordinary differential equations consists in their discretization. In place of the standard discretization based on the arithmetic progression, one can use a not less efficient q-discretization related to geometric progression [1]. This alternative method leads to q-difference equations which, in the limit $q \to 1$, correspond to the original differential equations. The theory of q-difference equations and related theory of q-special functions have a long story (see, for example [2]). During the last few decades they have been reviewed because of the great success of the theory of quantum groups.

The other crucial way of solving ordinary differential equations is based on the factorization method first used by Darboux [3]. We refer to [4] for an exhaustive presentation of the factorization method.

This work provides with an extension of the results recently published by Atakishiyev *et al* [5] on the continuous q-Hermite polynomials, which occupy the lowest level in the hierachy of ${}_4\phi_3$ polynomials with positive orthogonality measures and admit a factorized form $\mathcal{D}_x^q H_n(x|q) = q^{-n/2} H_n(x|q)$, $\mathcal{D}_x^q$ being some explicitly known q-difference operator (see equation (14) in [5]). We show that, for $0 < q < 1$, the continuous q-Jacobi polynomials $P_n^{(\alpha,\beta)}(x|q)$ satisfy the relation $\mathcal{D}_x^q(\alpha,\beta) P_n^{(\alpha,\beta)}(x|q) = (q^{-n/2} + q^{(n+\alpha+\beta+1)/2}) P_n^{(\alpha,\beta)}(x|q)$, where $\mathcal{D}_x^q(\alpha,\beta)$ is an explicit known q-difference operator.

The paper is organized as follows. In section 2, we give the factorization of the q-difference equations for the continuous q-Jacobi polynomials. Section 3 is devoted to some concluding remarks where we discussed the limit cases of parameters associated with the factorization, leading to some relevant classes of orthogonal polynomials.

2. Factorization of the continuous q-Jacobi polynomials

In this section, we study the Sturm–Liouville type of q-difference equation for the continuous q-Jacobi polynomials. We adopt the common conventions and notations on q-series. So, we always assume that $0 < q < 1$ and use the following notations of the q-shifted factorial:

$$(x;q)_0 = 1 \quad (x;q)_n = \prod_{j=1}^{n}(1-q^{j-1}x) \qquad n = 1, 2, \ldots, \infty \tag{1}$$

$$(x, y; q)_n = (x;q)_n (y;q)_n. \tag{2}$$

The basis hypergeometric series ${}_r\phi_s$ is defined as follows [6]:

$$ {}_r\phi_s\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}; q, x\right) \tag{3}$$

$$= \sum_{k=0}^{+\infty} \frac{(a_1;q)_k \ldots (a_r;q)_k}{(q;q)_k (b_1;q)_k \ldots (b_s;q)_k} [(-1)^k q^{k(k-1)/2}]^{1+s-r} x^k. \tag{4}$$

It is well known (see, for example, [7]) that for $\alpha \geqslant -1/2$ and $\beta \geqslant -1/2$, the continuous q-Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x|q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n} \times {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n+\alpha+\beta+1}, q^{\alpha/2+1/4}\,\mathrm{e}^{\mathrm{i}\theta}, q^{\alpha/2+1/4}\,\mathrm{e}^{-\mathrm{i}\theta} \\ q^{\alpha+1}, -q^{(\alpha+\beta+1)/2}, -q^{(\alpha+\beta+2)/2} \end{matrix}; q, q\right) \qquad x = \cos\theta \tag{5}$$

are orthogonal on the finite interval $-1 \leqslant x := \cos\theta \leqslant 1$:

$$\frac{1}{2\pi}\int_{-1}^{1} \tilde{w}(x; q^\alpha, q^\beta|q) P_n^{(\alpha,\beta)}(x|q) P_m^{(\alpha,\beta)}(x|q)\,\mathrm{d}x = \frac{(q^{(\alpha+\beta+2)/2}, q^{(\alpha+\beta+3)/2}; q)_\infty}{(q, q^{\alpha+1}, q^{\beta+1}, -q^{(\alpha+\beta+1)/2}, -q^{(\alpha+\beta+2)/2}; q)_\infty} \times \frac{(1-q^{\alpha+\beta+1})(q^{\alpha+1}, q^{\beta+1}, -q^{(\alpha+\beta+3)/2}; q)_n}{(1-q^{2n+\alpha+\beta+1})(q, q^{\alpha+\beta+1}, -q^{(\alpha+\beta+1)/2}; q)_n} q^{(\alpha+1/2)n}\delta_{nm} \tag{6}$$

with respect to the weight function

$$\tilde{w}(x; q^\alpha, q^\beta|q) = \frac{1}{\sin\theta} \frac{(\mathrm{e}^{2\mathrm{i}\theta}, \mathrm{e}^{-2\mathrm{i}\theta}; q)_\infty}{(a\,\mathrm{e}^{\mathrm{i}\theta}, b\,\mathrm{e}^{\mathrm{i}\theta}; q)_\infty (a\,\mathrm{e}^{-\mathrm{i}\theta}, b\,\mathrm{e}^{-\mathrm{i}\theta}; q)_\infty (c\,\mathrm{e}^{\mathrm{i}\theta}, d\,\mathrm{e}^{\mathrm{i}\theta}; q)_\infty (c\,\mathrm{e}^{-\mathrm{i}\theta}, d\,\mathrm{e}^{-\mathrm{i}\theta}; q)_\infty} \tag{7}$$

where

$$a = q^{\alpha/2+1/4}, \qquad b = q^{\alpha/2+3/4}, \qquad c = -q^{\beta/2+1/4}, \qquad d = -q^{\beta/2+3/4}. \tag{8}$$

These polynomials satisfy the q-difference equation

$$\begin{aligned} D_q\left[\tilde{w}(x;q^{\alpha+1},q^{\beta+1}|q)D_q P_n^{(\alpha,\beta)}(x|q)\right] \\ = \frac{4q(1-q^{-n})(1-q^{n+\alpha+\beta+1})}{(1-q)^2}\tilde{w}(x;q^{\alpha},q^{\beta}|q)P_n^{(\alpha,\beta)}(x|q) \end{aligned} \tag{9}$$

written on a self-adjoint form [9]. The D_q in (9) is the conventional notation for the Askey–Wilson divided-difference operator defined as

$$\begin{aligned} D_q f(x) &:= \frac{\delta_q f(x)}{\delta_q x} \\ \delta_q g(\mathrm{e}^{\mathrm{i}\theta}) &:= g(q^{1/2}\,\mathrm{e}^{\mathrm{i}\theta}) - g(q^{-1/2}\,\mathrm{e}^{\mathrm{i}\theta}), \qquad f(x)\equiv g(\mathrm{e}^{\mathrm{i}\theta}), \qquad x=\cos\theta. \end{aligned} \tag{10}$$

In terms of the shift operators with respect to the variable θ, the operator D_q can be explicitly expressed as

$$D_q f(x) = \frac{\sqrt{q}}{\mathrm{i}(1-q)}\frac{1}{\sin\theta}\left(\mathrm{e}^{\mathrm{i}\ln q^{1/2}\partial_\theta} - \mathrm{e}^{-\mathrm{i}\ln q^{1/2}\partial_\theta}\right)f(x) \qquad \partial_\theta = \frac{\mathrm{d}}{\mathrm{d}\theta}. \tag{11}$$

In order to write the q-difference equation for the continuous q-Jacobi polynomials in the factorized form, let us eliminate the weight function $\tilde{w}(x;q^{\alpha},q^{\beta}|q)$ from (9) using the relation

$$\begin{aligned} \exp(\pm\mathrm{i}\ln q^{1/2}\partial_\theta)\tilde{w}(x;q^{\alpha+1},q^{\beta+1}|q) \\ = -\frac{\mathrm{e}^{\pm2\mathrm{i}\theta}}{\sqrt{q}}(1-a\,\mathrm{e}^{\mp\mathrm{i}\theta})(1-b\,\mathrm{e}^{\mp\mathrm{i}\theta})(1-c\,\mathrm{e}^{\mp\mathrm{i}\theta})(1-d\,\mathrm{e}^{\mp\mathrm{i}\theta})\tilde{w}(x;q^{\alpha},q^{\beta}|q). \end{aligned} \tag{12}$$

The validity of (12) is straightforwardly checked upon using the explicit expression (7) and the identities

$$b = q^{1/2}a, \qquad d = q^{1/2}c. \tag{13}$$

Thus, combining (12) and (9) results in the following q-difference equation for the continuous q-Jacobi polynomials $P_n^{(\alpha,\beta)}(x|q)$, which does not explicitly contain the weight function $\tilde{w}(x;q^{\alpha},q^{\beta}|q)$

$$\begin{aligned} \frac{1}{2\mathrm{i}\sin\theta}\left[\frac{u(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{e}^{-\mathrm{i}\theta}}{1-q\,\mathrm{e}^{2\mathrm{i}\theta}}(1-\mathrm{e}^{-\mathrm{i}\ln q\partial_\theta}) + \frac{u(\mathrm{e}^{-\mathrm{i}\theta})\,\mathrm{e}^{\mathrm{i}\theta}}{1-q\,\mathrm{e}^{-2\mathrm{i}\theta}}(\mathrm{e}^{\mathrm{i}\ln q\partial_\theta}-1)\right]P_n^{(\alpha,\beta)}(x|q) \\ = (q^{-n}-1)(1-q^{n+\alpha+\beta+1})P_n^{(\alpha,\beta)}(x|q), \end{aligned} \tag{14}$$

where

$$u(\mathrm{e}^{\mathrm{i}\theta}) = (1-a\,\mathrm{e}^{\mathrm{i}\theta})(1-b\,\mathrm{e}^{\mathrm{i}\theta})(1-c\,\mathrm{e}^{\mathrm{i}\theta})(1-d\,\mathrm{e}^{\mathrm{i}\theta}).$$

In connection with equation (14), let us point out that Koornwinder [10] has recently studied in detail raising and lowering relations for the Askey–Wilson polynomials $p_n(x;a',b',c',d'|q)$. For $a'=a$, $b'=b$, $c'=c$ and $d'=d$, the Askey–Wilson polynomials reduce to the continuous q-Jacobi polynomials. One can readily check that (14) coincides with ‘ the second-order q-difference formula ’ (4.6) in [10], upon taking into account that variables z is equal to $\mathrm{e}^{\mathrm{i}\theta}$ and a, b, c, d are replaced by their values given in (8).

We are now in a position to show that (14) admits a factorization. Indeed, provided the trigonometric identities

$$\frac{\mathrm{e}^{\pm\mathrm{i}\theta}}{2\mathrm{i}\sin\theta} = \pm\frac{1}{1-\mathrm{e}^{\mp2\mathrm{i}\theta}} \tag{15}$$

the lhs of (14) can be expanded as

$$\frac{1}{2\mathrm{i}\sin\theta}\left[\frac{u(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{e}^{-\mathrm{i}\theta}}{1-q\,\mathrm{e}^{2\mathrm{i}\theta}}(1-\mathrm{e}^{-\mathrm{i}\ln q\partial_\theta})+\frac{u(\mathrm{e}^{-\mathrm{i}\theta})\,\mathrm{e}^{\mathrm{i}\theta}}{1-q\,\mathrm{e}^{-2\mathrm{i}\theta}}(\mathrm{e}^{\mathrm{i}\ln q\partial_\theta}-1)\right]P_n^{(\alpha,\beta)}(x|q)$$
$$=\left[\frac{(1-a\,\mathrm{e}^{-\mathrm{i}\theta})(1-c\,\mathrm{e}^{-\mathrm{i}\theta})}{(1-\mathrm{e}^{-2\mathrm{i}\theta})}\,\mathrm{e}^{\mathrm{i}\ln q^{1/2}\partial_\theta}\left(\frac{(1-a\,\mathrm{e}^{-\mathrm{i}\theta})(1-c\,\mathrm{e}^{-\mathrm{i}\theta})}{(1-\mathrm{e}^{-2\mathrm{i}\theta})}\,\mathrm{e}^{\mathrm{i}\ln q^{1/2}\partial_\theta}\right)\right.$$
$$+\frac{(1-a\,\mathrm{e}^{\mathrm{i}\theta})(1-c\,\mathrm{e}^{\mathrm{i}\theta})}{(1-\mathrm{e}^{2\mathrm{i}\theta})}\,\mathrm{e}^{-\mathrm{i}\ln q^{1/2}\partial_\theta}\left(\frac{(1-a\,\mathrm{e}^{\mathrm{i}\theta})(1-c\,\mathrm{e}^{\mathrm{i}\theta})}{(1-\mathrm{e}^{2\mathrm{i}\theta})}\,\mathrm{e}^{-\mathrm{i}\ln q^{1/2}\partial_\theta}\right)$$
$$\left.-\frac{u(\mathrm{e}^{\mathrm{i}\theta})}{(1-\mathrm{e}^{2\mathrm{i}\theta})(1-q\,\mathrm{e}^{2\mathrm{i}\theta})}-\frac{u(\mathrm{e}^{-\mathrm{i}\theta})}{(1-\mathrm{e}^{-2\mathrm{i}\theta})(1-q\,\mathrm{e}^{-2\mathrm{i}\theta})}\right]P_n^{(\alpha,\beta)}(x|q) \tag{16}$$

where use has been made of the identity

$$\frac{(1-b\,\mathrm{e}^{\mp\mathrm{i}\theta})(1-d\,\mathrm{e}^{\mp\mathrm{i}\theta})}{1-q\,\mathrm{e}^{\mp2\mathrm{i}\theta}}\,\mathrm{e}^{\pm\mathrm{i}\ln q^{1/2}\partial_\theta}=\mathrm{e}^{\pm\mathrm{i}\ln q^{1/2}\partial_\theta}\,\frac{(1-a\,\mathrm{e}^{\mp\mathrm{i}\theta})(1-c\,\mathrm{e}^{\mp\mathrm{i}\theta})}{1-\mathrm{e}^{\mp2\mathrm{i}\theta}}. \tag{17}$$

After tedious computation, taking into account (13), the rhs expression in square brackets in (16) can be rewritten as

$$\left(\mathcal{D}_x^q(\alpha,\beta)\right)^2+\frac{(-1+ac)(q-bd)}{q}. \tag{18}$$

Hence, (14) becomes

$$\left(\mathcal{D}_x^q(\alpha,\beta)\right)^2P_n^{(\alpha,\beta)}(x|q)=(q^{-n}+2q^{(\alpha+\beta+1)/2}+q^{n+\alpha+\beta+1})P_n^{(\alpha,\beta)}(x|q), \tag{19}$$

where the q-difference operator $\mathcal{D}_x^q(\alpha,\beta)$ is given by

$$\mathcal{D}_x^q(\alpha,\beta)=\frac{(1-a\,\mathrm{e}^{-\mathrm{i}\theta})(1-c\,\mathrm{e}^{-\mathrm{i}\theta})}{1-\mathrm{e}^{-2\mathrm{i}\theta}}\,\mathrm{e}^{\mathrm{i}\ln q^{1/2}\partial_\theta}+\frac{(1-a\,\mathrm{e}^{\mathrm{i}\theta})(1-c\,\mathrm{e}^{\mathrm{i}\theta})}{1-\mathrm{e}^{2\mathrm{i}\theta}}\,\mathrm{e}^{-\mathrm{i}\ln q^{1/2}\partial_\theta}. \tag{20}$$

Finally, taking into account that the factor on the rhs of (19) can be written as $q^{-n}+2q^{(\alpha+\beta+1)/2}+q^{n+\alpha+\beta+1}=(q^{-n/2}+q^{(n+\alpha+\beta+1)/2})^2$, one arrives at the following factorized form of (19):

$$\left(\mathcal{D}_x^q(\alpha,\beta)\right)^2P_n^{(\alpha,\beta)}(x|q)=(q^{-n/2}+q^{(n+\alpha+\beta+1)/2})^2P_n^{(\alpha,\beta)}(x|q). \tag{21}$$

Note that the operator $\left(\mathcal{D}_x^q(\alpha,\beta)\right)^2$ represents, as (21) implies, an unbounded operator on the Hilbert space $L^2(\mathbb{S}^1)$ with the scalar product

$$\langle g_1,g_2\rangle=\frac{1}{2\pi}\int_{-1}^{1}g_1(x)\overline{g_2(x)}\tilde{w}(x;q^\alpha,q^\beta|q)\,\mathrm{d}x \tag{22}$$

where the weight function $\tilde{w}(x;q^\alpha,q^\beta|q)$ is defined by (7). In view of (6) the polynomials

$$p_n(x)=(d_n(\alpha,\beta))^{-1/2}P_n^{(\alpha,\beta)}(x|q)\qquad n=0,1,2,\ldots, \tag{23}$$

where

$$d_n(\alpha,\beta)=\frac{(q^{(\alpha+\beta+2)/2},q^{(\alpha+\beta+3)/2};q)_\infty}{(q,q^{\alpha+1},q^{\beta+1},-q^{(\alpha+\beta+1)/2},-q^{(\alpha+\beta+2)/2};q)_\infty}$$
$$\times\frac{(1-q^{\alpha+\beta+1})(q^{\alpha+1},q^{\beta+1},-q^{(\alpha+\beta+3)/2};q)_n}{(1-q^{2n+\alpha+\beta+1})(q,q^{\alpha+\beta+1},-q^{(\alpha+\beta+1)/2};q)_n}q^{(\alpha+1/2)n} \tag{24}$$

coçnstitute an orthonormal basis in this space such that we will have the relation $\left(\mathcal{D}_x^q(\alpha,\beta)\right)^2p_n(x)=(q^{-n/2}+q^{(n+\alpha+\beta+1)/2})^2p_n(x)$. In particular, the operator $\left(\mathcal{D}_x^q(\alpha,\beta)\right)^2$ is defined on the linear span $\mathcal{H}$ of the basis function p_n, which is everywhere dense in $L^2(\mathbb{S}^1)$. We close $\left(\mathcal{D}_x^q(\alpha,\beta)\right)^2$ with respect to the scalar product (22). Since $\left(\mathcal{D}_x^q(\alpha,\beta)\right)^2$ is diagonal with respect to the orthonormal basis $p_n(x)$, $n = 0, 1, 2, \ldots$, its closure $\overline{\left(\mathcal{D}_x^q(\alpha, \beta)\right)^2}$ is a self-adjoint operator (see [8], chapter 6). We can take a square root of the operator $\overline{\left(\mathcal{D}_x^q(\alpha, \beta)\right)^2}$. This square root is a self-adjoint operator too and has the same eigenfunctions as the operator $\overline{\left(\mathcal{D}_x^q(\alpha, \beta)\right)^2}$ does. We denote this operator by $\overline{\mathcal{D}_x^q(\alpha, \beta)}$. It is evident that, on the subspace $\mathcal{H}$, the operator $\overline{\mathcal{D}_x^q(\alpha, \beta)}$ coincides with the operator $\mathcal{D}_x^q(\alpha, \beta)$. Hence, the operator $\mathcal{D}_x^q(\alpha, \beta)$ is a well-defined operator on the Hilbert space $L^2(\mathbb{S}^1)$ with everywhere dense subspace of the definition. Moreover, according to the definition of a self-adjoint operator (see [8], chapter 6), we have

$$\overline{\mathcal{D}_x^q(\alpha, \beta)}\, p_n(x) = \left(q^{-n/2} + q^{(n+\alpha+\beta+1)/2}\right) p_n(x). \tag{25}$$

This means that the continuous q-Jacobi polynomials $P_n^{(\alpha,\beta)}(x|q)$ are in fact governed by a simpler q-difference equation

$$\mathcal{D}_x^q(\alpha, \beta) P_n^{(\alpha,\beta)}(x|q) = \left(q^{-n/2} + q^{(n+\alpha+\beta+1)/2}\right) P_n^{(\alpha,\beta)}(x|q). \tag{26}$$

Observe that the q-difference operator $\mathcal{D}_x^q(\alpha, \beta)$ in (26) may be expressed in terms of the Askey–Wilson operator D_q defined in (10), as

$$\mathcal{D}_x^q(\alpha, \beta) = (1 - ac)\mathcal{A}_q + \frac{1-q}{2\sqrt{q}}\left[(1 + ac)x - (a + c)\right] D_q \tag{27}$$

where $\mathcal{A}_q$ is the so-called *averaging difference operator* defined in [7] as

$$(\mathcal{A}_q f)(x) = \tfrac{1}{2}\left(\mathrm{e}^{\mathrm{i}\ln q^{1/2}\partial_\theta} + \mathrm{e}^{-\mathrm{i}\ln q^{1/2}\partial_\theta}\right) f(x) \equiv \cos(\ln q^{1/2}\partial_\theta) f(x). \tag{28}$$

The q-difference equation (26) is consistent with the generating function [6]

$$\begin{aligned}
&{}_2\phi_1\left(\begin{matrix} a\,\mathrm{e}^{\mathrm{i}\theta}, d\,\mathrm{e}^{\mathrm{i}\theta} \\ ad \end{matrix}\; q; \mathrm{e}^{-\mathrm{i}\theta} t\right) {}_2\phi_1\left(\begin{matrix} b\,\mathrm{e}^{-\mathrm{i}\theta}, c\,\mathrm{e}^{-\mathrm{i}\theta} \\ bc \end{matrix}\; q; \mathrm{e}^{\mathrm{i}\theta} t\right) \\
&\quad = \sum_{n=0}^{\infty} \frac{(-q^{(\alpha+\beta+1)/2}; q)_n}{(-q^{(\alpha+\beta+2)/2}; q)_n} \frac{P_n^{(\alpha,\beta)}(x|q)}{q^{(2\alpha+1)n/4}} t^n
\end{aligned} \tag{29}$$

for the continuous q-Jacobi polynomials $P_n^{(\alpha,\beta)}(x|q)$.

Indeed, by applying the q-difference operator $\mathcal{D}_x^q(\alpha, \beta)$ defined in (20), and by using (13) and the property [6]

$${}_2\phi_1\left(\begin{matrix} e, f \\ g \end{matrix}\; q; y\right) = \frac{(efg^{-1}y; q)_\infty}{(y; q)_\infty} {}_2\phi_1\left(\begin{matrix} e^{-1}g, f^{-1}g \\ g \end{matrix}\; q; efg^{-1}y\right) \tag{30}$$

it is straightfoward to verify that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(-q^{(\alpha+\beta+1)/2}; q)_n}{(-q^{(\alpha+\beta+2)/2}; q)_n} \frac{\mathcal{D}_x^q(\alpha, \beta)(P_n^{(\alpha,\beta)}(x|q))}{q^{(2\alpha+1)n/4}} t^n \\
&\quad = \mathcal{D}_x^q(\alpha, \beta)\left({}_2\phi_1\left(\begin{matrix} a\,\mathrm{e}^{\mathrm{i}\theta}, d\,\mathrm{e}^{\mathrm{i}\theta} \\ ad \end{matrix}\; q; \mathrm{e}^{-\mathrm{i}\theta} t\right) {}_2\phi_1\left(\begin{matrix} b\,\mathrm{e}^{-\mathrm{i}\theta}, c\,\mathrm{e}^{-\mathrm{i}\theta} \\ bc \end{matrix}\; q; \mathrm{e}^{\mathrm{i}\theta} t\right)\right) \\
&\quad = {}_2\phi_1\left(\begin{matrix} a\,\mathrm{e}^{\mathrm{i}\theta}, d\,\mathrm{e}^{\mathrm{i}\theta} \\ ad \end{matrix}\; q; q^{-1/2}\mathrm{e}^{-\mathrm{i}\theta} t\right) {}_2\phi_1\left(\begin{matrix} b\,\mathrm{e}^{-\mathrm{i}\theta}, c\,\mathrm{e}^{-\mathrm{i}\theta} \\ bc \end{matrix}\; q; q^{-1/2}\mathrm{e}^{\mathrm{i}\theta} t\right)
\end{aligned}$$

$$- ac\, {}_2\phi_1\left(\begin{matrix} a\,\mathrm{e}^{\mathrm{i}\theta}, d\,\mathrm{e}^{\mathrm{i}\theta} \\ ad \end{matrix} ; q; q^{1/2}\,\mathrm{e}^{-\mathrm{i}\theta} t\right) {}_2\phi_1\left(\begin{matrix} b\,\mathrm{e}^{-\mathrm{i}\theta}, c\,\mathrm{e}^{-\mathrm{i}\theta} \\ bc \end{matrix} ; q; q^{1/2}\,\mathrm{e}^{\mathrm{i}\theta} t\right)$$

$$= \sum_{n=0}^{\infty} \frac{(-q^{(\alpha+\beta+1)/2}; q)_n}{(-q^{(\alpha+\beta+2)/2}; q)_n} \frac{P_n^{(\alpha,\beta)}(x|q)}{q^{(2\alpha+1)n/4}} (q^{-n/2} + q^{(\alpha+\beta+1)/2} q^{n/2}) t^n. \tag{31}$$

Equating coefficients of like powers of t on the extremal sides of (31), one completes the another proof of equation (26).

In the limit $q \to 1$, the continuous q-Jacobi polynomials reduce to the Jacobi polynomials [6]:

$$\lim_{q\to 1} P_n^{(\alpha,\beta)}(x|q) = P_n^{(\alpha,\beta)}(x). \tag{32}$$

Then, in the limit $q \to 1$, the q-difference equation (9) reduces to the following second-order differential equation

$$\left\{(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + [\beta - \alpha - (\alpha+\beta+2)x]\frac{\mathrm{d}}{\mathrm{d}x} + n(n+\alpha+\beta+1)\right\} P_n^{(\alpha,\beta)}(x) = 0. \tag{33}$$

This fact can be also expressed as the following limit property of the q-difference operator $\mathcal{D}_x^q(\alpha,\beta)$ in (20):

$$\lim_{q\to 1}\left\{\frac{2}{(1-q)^2}\left[\left(1+q^{(\alpha+\beta+1)/2}\right)I - \mathcal{D}_x^q(\alpha,\beta)\right]\right\}$$
$$= \frac{1}{2}\left\{[\beta-\alpha-(\alpha+\beta+2)x]\frac{\mathrm{d}}{\mathrm{d}x} + (1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right\}, \tag{34}$$

where I is the identity operator.

3. Concluding remarks

The q-difference equation (26) for the continuous q-Jacobi polynomials, derived in the previous section, does actually contain some special and limit cases of the parameters α and β, which correspond to well-known families of q-polynomials. In the case $\alpha = \beta$, the continuous q-Jacobi polynomials $P^{(\alpha,\beta)}(x|q)$ reduce to, up to a normalization factor, the continuous q-ultraspherical polynomials $C_n(x;\beta'|q)$ where $\beta' = q^{\alpha+1/2}$. Hence, the above results appear as an extension of the work by Area et *al* [11]. One can then easily deduce q-Hermite and q-Legendre polynomials as done in [11].

Finally, it is noteworthy that, in the limit case $\beta \to \infty$, the continuous q-Jacobi polynomials reduce to continuous q-Laguerre polynomials $P_n^{(\alpha)}(x|q)$.

This work was mainly focused on the factorization of q-continuous Jacobi polynomials. We have shown that these polynomials admit a factorization of the form given by Atakishiyev *et al* (see equation (14) in [5]). The results obtained here essentially come from parameter relations (13) which are not satisfied by all classes of Askey–Wilson polynomials. Hence, the search for a unified framework of extending the above formalism to the whole Askey tableau remains a real challenge. Such a goal deserves serious consideration.

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